Topology and its Applications 157 (2010) 536–547

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

The terminal hyperspace of homogeneous continua

Janusz R. Prajs [a,b,*,1]

[a] California State University Sacramento, Department of Mathematics and Statistics, 6000 J Street, Sacramento, CA 95819, USA
[b] University of Opole, Institute of Mathematics, Ul. Oleska 48, 45-052 Opole, Poland

ARTICLE INFO

Article history:
Received 7 February 2009
Received in revised form 22 October 2009
Accepted 22 October 2009

MSC:
primary 54F15
secondary 54F50

Keywords:
Continuum
Decomposition
Homogeneous
Indecomposable
Intrinsic
Terminal
Tree-like

ABSTRACT

We investigate the structure of the collection of terminal subcontinua in homogeneous continua. The main result is a reduction of this structure to six specific types. Three of these types are of one-dimensional spaces, and examples representing these types are known. It is not known whether higher dimensional examples having non-trivial terminal subcontinua and representing the three remaining types exist.

In 1948, R.H. Bing made a major breakthrough [2] in the study of homogeneous continua by showing the homogeneity of Moise' *pseudo-arc* [18]. The pseudo-arc is *hereditarily indecomposable*; in fact it is homeomorphic to earlier defined *Knaster's continuum* [3,10]. By this example subcontinua later called *terminal* were introduced to the study of homogeneous spaces. Though terminal continua in homogeneous spaces may seem counterintuitive, every subcontinuum of a hereditarily indecomposable continuum, such as the pseudo-arc, is terminal. In 1955, F.B. Jones published [8] his aposyndetic decomposition theorem for homogeneous continua. The elements of this decomposition are terminal. The *circle of pseudo-arcs* defined in 1959 by Bing and Jones [4] is a homogeneous continuum having the aposyndetic decomposition non-trivial, and all sufficiently small subcontinua terminal. Terminal subcontinua of homogeneous continua became a subject of extensive study. James T. Rogers showed [24] that homogeneous continua having all subcontinua terminal are tree-like. Conversely, the author showed, in a joint paper with Paweł Krupski [11], that homogeneous tree-like continua are hereditarily indecomposable, that is, they have all subcontinua terminal. A large class of new examples of homogeneous continua having all sufficiently small subcontinua terminal was defined by Wayne Lewis [14]. Rogers [25–27] studied continuous decompositions of homogeneous continua into terminal subcontinua. Terminal subcontinua of homogeneous spaces were also the subject of study by Tadeusz Maćkowiak and Edward Tymchatyn [16], Maćkowiak [15], Charles L. Hagopian [7], Zhou Youcheng and Lin Shou [31], and others. Recently, Rogers [28] has shown that the quotient space of a homogeneous continuum under

* Address for correspondence: California State University Sacramento, Department of Mathematics and Statistics, 6000 J Street, Sacramento, CA 95819, USA.
E-mail address: prajs@csus.edu.
1 The author was supported in part by the Research and Creative Activity Award 2008 at California State University, Sacramento.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.10.011

the aposyndetic decomposition with non-trivial fibers has dimension at most 1. This new result is an essential tool in this paper.

The study presented in this paper was initiated in 1999, in collaboration with Carl Seaquist, when the author was a visiting faculty at Texas Tech University in Lubbock. At this early stage we were looking for new properties of the set of terminal subcontinua in homogeneous spaces, and asking questions such as: Is the set of the components of the hyperspace of terminal subcontinua always finite? If not, is it countable? Some of the concepts used in this paper, such as a *locally minimal terminal subcontinuum* and an *intrinsic decomposition*, appeared during these discussions (under different names). This project was abandoned when the author left Texas Tech in 2000. The results presented in this paper are new but the significance of the early discussions and observations made in collaboration with Carl Seaquist should be acknowledged.

In this paper we answer many questions earlier asked in Lubbock. For instance, we show that the terminal hyperspace of a homogeneous continuum can have at most three components. Of these three components only one can have two different members Y_1 and Y_2 such that $Y_1 \subset Y_2$. Moreover, in this last case there is a unique order arc of terminal continua from Y_1 to Y_2. The study presented in this paper can be viewed as a step towards solving the following major open problem (compare related [13, Problems 87, 92 and 93]):

Question 1. If X is a homogeneous continuum and Y its proper terminal subcontinuum, is Y at most one-dimensional?

The main result asserts that the structure of the terminal hyperspace of a homogeneous continuum must represent one of the six types schematically pictured in Fig. 1 in the end of the paper. Each known example has type (a), (b) or (c). If a homogeneous continuum of type (d), (e) or (f) exists, Question 1 will be answered in the negative.

The tools developed in this paper can also be used in the future study of other classes of subcontinua of homogeneous continua such as *semi-terminal* ones [22,23].

1. Preliminaries

In this paper, all spaces are metric and mappings continuous. A space X is called *homogeneous* if for every $x, y \in X$ there is a homeomorphism $h : X \to X$ such that $h(x) = y$. An *ε-map* is a map $f : X \to Y$ between spaces X and Y such that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in f(X)$. If X is a space, 2^X denotes the collection of non-empty compact subsets of X, and $C(X)$, the collection of subcontinua of X, both equipped with the Hausdorff metric. For a compact space X, whenever in the paper we refer to a "closed collection of continua" or "closed collection of closed sets" we mean closed subsets of $C(X)$ or 2^X. Since 2^X and $C(X)$ are compact if X is, such collections are always compact. If X is compact, $H(X)$ stands for the group of self-homeomorphisms of X with the *sup* metric.

A one-dimensional continuum is called a *curve*. A *Kelley continuum* [9] is a continuum X such that for every $K \in C(X)$, every $p \in K$ and every sequence $\{p_n\} \subset X$ converging to p, there are continua K_n converging to K in $C(X)$ with $p_n \in K_n$ for each n. Each homogeneous continuum is Kelley [30].

If X is a homogeneous compact space, then for every positive ε there is a number δ, called an *Effros number* for ε, such that for each $x, y \in X$ with $d(x, y) < \delta$, there is some homeomorphism $h \in H(X)$ such that $h(x) = y$ and $d(z, h(z)) < \varepsilon$ for each $z \in X$. This is called the *Effros theorem*. It follows from the more general statement that for each $x \in X$, the evaluation map, $h \mapsto hx$, from the homeomorphism group $H(X)$ onto X is open. The latter follows from [6, Theorem 2]. (See also [29, Theorem 3.1].) By [5, Theorem 3.3], we have a similar result for Borel subgroups of $H(X)$.

2. Basic concepts

If X is a topological space, and $\mathcal{P}$ a collection of non-empty subsets of X, we say that $\mathcal{P}$ is an *intrinsic collection* provided $h(P) \in \mathcal{P}$ for every self-homeomorphism $h : X \to X$ of X and $P \in \mathcal{P}$. If $\mathcal{P}$ is intrinsic in X and partitions X, then other authors often call such $\mathcal{P}$ "a partition (or decomposition) respected by the group of self-homeomorphisms of X." Here we call such $\mathcal{P}$ an *intrinsic partition* or an *intrinsic decomposition* of X. A partition $\mathcal{P}$ of X is called *homogeneous* provided for every $x, y \in X$ there is a homeomorphism $h : X \to X$ such that $h(x) = y$ and $h(P) \in \mathcal{P}$ for each $P \in \mathcal{P}$. In other words, the partition $\mathcal{P}$ is homogeneous if and only if the group of self-homeomorphisms of X that respect $\mathcal{P}$ acts transitively on X. It is easy to see that upper semi-continuous homogeneous decompositions of compact spaces into closed subsets have both the members and quotient spaces homogeneous. Clearly, only homogeneous spaces can have homogeneous partitions. An intrinsic partition of a homogeneous space is always homogeneous. If $\mathbb{S}^1$ is the unit circle, and $X = \mathbb{S}^1 \times \mathbb{S}^1$ the torus, the decomposition of X into the circles $\mathbb{S}^1 \times \{s\}$ is homogeneous but not intrinsic.

Definition 2.1. A set Y in a space X is called *fastened* provided that $h(Y) = Y$ for every homeomorphism $h : X \to X$ such that $h(Y) \cap Y \neq \emptyset$.

We observe the following.

Proposition 2.2. *The members of an intrinsic decomposition of a space X are fastened in X.*

It is useful to consider collections that may not necessarily be intrinsic but are preserved by the homeomorphisms that are sufficiently near to the identity.

Definition 2.3. If $\mathcal{P}$ is a collection of subsets of a compact space X such that for some neighborhood U of the identity in $H(X)$ we have $h(P) \in \mathcal{P}$ for every $P \in \mathcal{P}$ and every homeomorphism $h \in U$, then $\mathcal{P}$ is said to be a *partly intrinsic collection of* X. If, additionally, $\mathcal{P}$ partitions X, then we call $\mathcal{P}$ a *partly intrinsic partition* (or *decomposition*) of X.

If X is either the pseudo-arc $\mathbb{P}$, or the Menger curve $\mathbb{M}$, then every self-homeomorphism of $X \times X$ is the product of self-homeomorphisms of X, possibly, composed with the interchange of the coordinates ([1], [12, Theorem 2]). From these results it follows that if $X = \mathbb{P}$ or $X = \mathbb{M}$, then the decomposition $\mathcal{D}$ of $X \times X$ into the continua $X \times \{x\}$ is partly intrinsic but not intrinsic. Indeed, the products of self-homeomorphisms of X act transitively on $X \times X$ and form a neighborhood of the identity in $H(X)$. Therefore $\mathcal{D}$ is partly intrinsic. The homeomorphism that interchanges the coordinates does not respect $\mathcal{D}$, and thus $\mathcal{D}$ is not intrinsic.

Proposition 2.4. *A partly intrinsic decomposition of a homogeneous continuum into closed sets is homogeneous and continuous.*

Proof. Let $\mathcal{D}$ be a partly intrinsic decomposition of a homogeneous continuum X, and G the group of homeomorphisms $h: X \to X$ that respect $\mathcal{D}$. Let U be a neighborhood of the identity contained in G. By the Effros theorem the set $V_p = \{h(p) \mid h \in U\}$ is a neighborhood of p. The set V_p is contained in the orbit of p under G, and thus this orbit is open. The orbits under G are mutually disjoint. Therefore, there is only one such orbit by the connectedness of X. Hence $\mathcal{D}$ is a homogeneous decomposition of X. If $\lim p_n = p_0$ and $D_0, D_1, D_2, \ldots$ are the members of $\mathcal{D}$ containing the points $p_0, p_1, p_2, \ldots$, respectively, then, by the Effros theorem, there are homeomorphisms $h_n: X \to X$ converging to the identity such that $h_n(p_0) = p_n$. Thus $h_n \in U \subset G$ for almost all n. Consequently $h_n(D_0) = D_n$ for almost all n. Therefore, D_n converges to D_0 in the sense of the Hausdorff distance. Hence $\mathcal{D}$ is continuous. $\square$

Definition 2.5. A set Y in a compact space X is called *partly fastened* provided there is a neighborhood U of the identity in $H(X)$ such that $h(Y) = Y$ for every homeomorphism $h \in U$ satisfying $h(Y) \cap Y \neq \emptyset$.

Proposition 2.6. *Let Y be a partly fastened subset of a compact space X. Then for some neighborhood $U_Y \subset H(X)$ of the identity, and every $h, g \in U_Y$ such that $h(Y) \cap g(Y) \neq \emptyset$ we have $h(Y) = g(Y)$.*

Proof. Otherwise we would have two sequences $\{h_n\}, \{g_n\} \subset H(X)$ converging to the identity such that $h_n(Y) \cap g_n(Y) \neq \emptyset$ and $h_n(Y) \neq g_n(Y)$. The sequence of homeomorphisms $f_n = h_n^{-1} \circ g_n$ converges to the identity, $f_n(Y) \cap Y \neq \emptyset$ and $f_n(Y) \neq Y$, and thus Y is not partly fastened, a contradiction. $\square$

It is easy to verify the following.

Proposition 2.7. *The members of a partly intrinsic decomposition of a compact space X are partly fastened in X.*

Proposition 2.8. *If Y is a partly fastened subcontinuum of a homogeneous compact space X, then the group of homeomorphisms $h: X \to X$ such that $h(Y) = Y$ acts transitively on Y. In particular, Y is homogeneous.*

Proof. Let G be the group of homeomorphisms $h: X \to X$ such that $h(Y) = Y$, and U the neighborhood of the identity satisfying $h \in G$ whenever $h \in U$ and $h(Y) \cap Y \neq \emptyset$. Given $p \in Y$, by the Effros theorem $V_p = \{h(p) \mid h \in U\}$ is a neighborhood of p. Moreover, if $h(p) \in Y$ for some $h \in U$, then $h \in G$. Therefore $V_p \cap Y$ is in the orbit of p under G. Thus the orbits of points in Y under G are open in Y. Consequently, there is only one such orbit by the connectedness of Y. Hence G acts transitively on Y. $\square$

Definition 2.9. Two collections $\mathcal{U}$ and $\mathcal{V}$ of compact subsets of spaces X and Y, respectively, are called *isomorphic* provided there is a bijection $f: \mathcal{U} \to \mathcal{V}$ such that f is a homeomorphism between $\mathcal{U}$ and $\mathcal{V}$ with respect to the Hausdorf metric, and, $U_1 \subset U_2$ if and only if $f(U_1) \subset f(U_2)$ for every $U_1, U_2 \in \mathcal{U}$. We call the map f an *isomorphism* between $\mathcal{U}$ and $\mathcal{V}$.

A continuum T in a space X is said to be *terminal* if for every continuum L in X intersecting T either $T \subset L$ or $L \subset T$. If X is a continuum, $\mathcal{T}(X)$ denotes the collection of terminal subcontinua of X. We equip $\mathcal{T}(X)$ with the topology induced by the Hausdorff distance, and call $\mathcal{T}(X)$ the *terminal hyperspace of* X. It is known that $\mathcal{T}(X)$ is closed in $C(X)$ if X is Kelley. If X is a continuum, clearly, the singletons $\{x\}$, for $x \in X$, and the whole space X, are members of $\mathcal{T}(X)$. For a continuum X, the symbol $\mathcal{T}_0(X)$ denotes the component of $\mathcal{T}(X)$ containing all singletons, and $\mathcal{T}_1(X)$, the component of $\mathcal{T}(X)$ having X as a member.

An upper semi-continuous decomposition of a space X into its terminal subcontinua is called *atomic*, and the corresponding quotient map is also called *atomic*. The following two propositions are known and easy to prove.

Proposition 2.10. *If $f : X \to Y$ an atomic map from a continuum X onto Y, then a continuum $T \subset Y$ is terminal in Y if and only if $f^{-1}(T)$ is terminal in X.*

Proposition 2.11. *If $f : X \to Y$ an atomic map from a continuum X onto a non-degenerate continuum Y, then X is indecomposable if and only if Y is.*

If an atomic decomposition $\mathcal{D}$ of a compact space X is a compact collection, then $\mathcal{D}$ is continuous, and the corresponding quotient map, open. Such decompositions are of particular interest to this study. If $\mathcal{D}$ is a continuous atomic decomposition of a continuum X with the quotient map $q : X \to Y = X/\mathcal{D}$, there are only two types of subcontinua of X: (i) properly contained in some member of $\mathcal{D}$, and (ii) saturated with respect to $\mathcal{D}$, that is, the unions of some members of $\mathcal{D}$. In this case, the class $\mathcal{C}$ of subcontinua saturated with respect to $\mathcal{D}$ is closed, and the induced map $K \mapsto q(K)$ defines an isomorphism from $\mathcal{C}$ to $C(Y)$. Using these observations and Proposition 2.10 the reader can easily verify the following proposition.

Proposition 2.12. *Let $\mathcal{D}$ be a continuous, atomic decomposition of a continuum X with the quotient map $q : X \to Y = X/\mathcal{D}$, and $\mathcal{T}_{\mathcal{D}}(X)$ the collection of the members of $\mathcal{T}(X)$ saturated with respect to $\mathcal{D}$. Then the assignment $T \mapsto q(T)$ defines an isomorphism between $\mathcal{T}_{\mathcal{D}}(X)$ and $\mathcal{T}(Y)$. The extended assignment $\mathcal{T}(X) \ni T \mapsto q(T) \in \mathcal{T}(Y)$ defines a map congruent to a retraction from $\mathcal{T}(X)$ to $\mathcal{T}_{\mathcal{D}}(X)$, which preserves inclusion.*

Another important and known observation, which is also expressed in Proposition 2.13 below, is that *a subcontinuum K of a terminal continuum Y in a space X is terminal in X if and only if K is terminal in Y*. This observation is very intuitive, and we apply it frequently without making formal reference to Proposition 2.13.

Proposition 2.13. *If Y is a terminal subcontinuum of a continuum X, then $\mathcal{T}(Y) = \mathcal{T}(X) \cap C(Y)$.*

The first part of the following proposition is know, and the proof of the second one is left to the reader. Both parts can easily be shown using the observation that, by its homogeneity, X is Kelley [30].

Proposition 2.14. *If Y_n is a sequence of terminal subcontinua of a homogeneous continuum X converging to a continuum $Y \subset X$, then Y is terminal and $C(Y_n)$ converges to $C(Y)$ in the Hausdorff metric of $C(C(X))$.*

For a homogeneous continuum X, there are two important atomic, intrinsic decompositions of X: (i) the *aposyndetic decomposition*, and (ii) the *terminal decomposition*. The aposyndetic decomposition was defined by Jones [8], and the terminal decomposition is due to Rogers [27]. Here we present a modified but equivalent definition of the terminal decomposition of a homogeneous continuum X. For $x \in X$, let $\mathrm{Tc}(x)$ be the union of proper terminal subcontinua of X containing x. We call $\mathrm{Tc}(x)$ the *terminal composant of X determined by x*. Since the singleton $\{x\}$ is terminal, $\mathrm{Tc}(x) \neq \emptyset$ whenever X is non-degenerate, and the terminal composants partition such X. The collection $\Phi(X)$ of the minimal continua containing at least one terminal composant of X is an atomic, intrinsic decomposition of X, which we call the *terminal decomposition of X*.

The following theorem is a combination and a generalization of two theorems of Rogers [26,28]. Rogers' results were proved for intrinsic decompositions. However, if a continuous decomposition is homogeneous, then essentially the same tools are available, including the Effros theorem. Indeed, the original proofs of Rogers from [26] and [28] can be reproduced with the only modification that the group $H_{\mathcal{D}}(X)$ of homeomorphisms $h : X \to X$ that respect $\mathcal{D}$ replaces the group $H(X)$ of all self-homeomorphisms of X. Note that $H_{\mathcal{D}}(X)$ is closed in $H(X)$ and see [5, Theorem 3.3] to verify the Effros theorem for $H_{\mathcal{D}}(X)$.

Theorem 2.15 *(J.T. Rogers). Let X be a continuum, and $\mathcal{D}$ a continuous, atomic, homogeneous decomposition of X into proper, non-degenerate subcontinua. Then the members of $\mathcal{D}$ are cell-like, indecomposable continua, and the quotient space $X/\mathcal{D}$ is one-dimensional.*

3. Intrinsic collections of closed sets

In this section we study general properties of intrinsic and partly intrinsic collections of closed sets in homogeneous continua. The results of this study are used in the next section to investigate terminal continua. In the future, these results can also be applied in the study of other classes of subcontinua, such as semi-terminal ones [22,23].

Proposition 3.1. *If X is a homogeneous continuum, $\mathcal{F}$ a closed, non-empty, partly intrinsic collection of non-empty closed subsets of X, and $\mathcal{K}$ a component of $\mathcal{F}$, then $\bigcup \mathcal{K} = X$.*

Proof. Let U be a neighborhood of the identity in $H(X)$ composed of the self-homeomorphisms of X that respect $\mathcal{F}$. Let $\mathcal{M}$ be a non-empty open-closed collection in $\mathcal{F}$. Clearly, $\mathcal{M}^* = \bigcup \mathcal{M}$ is a non-empty closed subset of X. Suppose $\mathcal{M}^* \neq X$, and let $p \in \operatorname{bd} \mathcal{M}^*$. Then $p \in M$ for some $M \in \mathcal{M}$, and there is a sequence $\{p_n\} \subset X - \mathcal{M}^*$ converging to p. By the Effros theorem there are homeomorphisms $h_n : X \to X$ converging to the identity such that $h_n(p) = p_n$. Since $\mathcal{M}$ is open in $\mathcal{F}$, and $h_n \in U$ for almost all n, eventually $h_n(M) \in \mathcal{M}$ and $h_n(p) = p_n \in \mathcal{M}^*$, a contradiction.

We have shown that $\bigcup \mathcal{M} = X$ for every non-empty, open-closed collection $\mathcal{M}$ of members of $\mathcal{F}$. Since each component $\mathcal{K}$ of $\mathcal{F}$ is the intersection of a nested sequence of such collections, it follows $\bigcup \mathcal{K} = X$. $\square$

Consider the following property of a collection $\mathcal{F}$ of sets in a space X:

$$\text{For every } F_1, F_2 \in \mathcal{F} \text{ such that } F_1 \cap F_2 \neq \emptyset, \text{ either } F_1 \subset F_2 \text{ or } F_2 \subset F_1. \tag{1}$$

Here we study closed collections of subcontinua satisfying property (1) in homogeneous continua. In the next section we apply these results to the terminal hyperspace of homogeneous continua.

Proposition 3.2. *Let X be a homogeneous continuum, and $\mathcal{F}$ a closed, non-empty, partly intrinsic collection of continua in X satisfying property* (1). *If $\mathcal{K}$ and $\mathcal{L}$ are components of $\mathcal{F}$, and $K \subset L$ for some $K \in \mathcal{K}$ and $L \in \mathcal{L}$, then every member of $\mathcal{K}$ is contained in some member of $\mathcal{L}$.*

Proof. If $\mathcal{K} = \mathcal{L}$, the conclusion holds. Assume $\mathcal{K} \neq \mathcal{L}$. Let U be a neighborhood of the identity in $H(X)$ composed of the self-homeomorphisms of X that respect $\mathcal{F}$. Let $\mathcal{L}_n$ be open-closed, nested collections in $\mathcal{F}$ such that $\mathcal{L} = \bigcap_n \mathcal{L}_n$. Since $\mathcal{K} \cap \mathcal{L} = \emptyset$, without loss of generality assume $\mathcal{K} \cap \mathcal{L}_n = \emptyset$ for each n. Let $\mathcal{M}_n$ be the set of continua $K' \in \mathcal{K}$ such that there is a continuum L' in $\mathcal{L}_n$ with $K' \subset L'$. We have $K \in \mathcal{M}_n$ for each n, and thus the collections $\mathcal{M}_n$ are non-empty. Since the collections $\mathcal{K}$ and $\mathcal{L}_n$ are closed, and the relation of inclusion is closed in $C(X) \times C(X)$, it follows that $\mathcal{M}_n$'s are closed collections as well.

Suppose, for some fixed n and $K_0 \in \mathcal{M}_n$, a sequence $\{K_m\} \subset \mathcal{K}$ converges to K_0. By assumption, there is $L_0 \in \mathcal{L}_n$ such that $K_0 \subset L_0$. This last inclusion must be proper because $\mathcal{K} \cap \mathcal{L}_n = \emptyset$. Choose a sequence $\{p_m\} \subset X$ converging to a point $p \in K_0$ such that $p_m \in K_m$. By the Effros theorem, there are homeomorphisms $h_m : X \to X$ converging to the identity such that $h_m(p) = p_m$, and $h_m \in U$ for almost all m. The continua $h_m^{-1}(K_m)$ intersect K_0, limit on K_0, and K_0 is properly contained in L_0. Since both $h_m^{-1}(K_m)$ and L_0 are in $\mathcal{F}$, and $\mathcal{F}$ satisfies property (1), it follows that $h_m^{-1}(K_m) \subset L_0$ for almost all m. Thus $K_m \subset h_m(L_0)$ for almost all m. The collection $\mathcal{L}_n$ is open in $\mathcal{F}$, and thus $h_m(L_0) \in \mathcal{L}_n$ for almost all m. Consequently, $K_m \in \mathcal{M}_n$ for such m's. Hence $\mathcal{M}_n$ is an open collection in $\mathcal{K}$.

We have shown that $\mathcal{M}_n$'s are open-closed, non-empty subcollections of $\mathcal{K}$. Since $\mathcal{K}$ is connected, $\mathcal{M}_n = \mathcal{K}$ for each n. Therefore, for each $K' \in \mathcal{K}$ and each n, there is an $L_n \in \mathcal{L}_n$ such that $K' \subset L_n$. The limit of a convergent subsequence of $\{L_n\}$ is in $\mathcal{L}$ and contains K'. The proof is complete. $\square$

If $\mathcal{A}$ and $\mathcal{B}$ are collections of sets, we write $\mathcal{A} \prec \mathcal{B}$ provided for each $A \in \mathcal{A}$:

(i) A is contained in some member of $\mathcal{B}$; and
(ii) no member of $\mathcal{B}$ is contained in A.

Corollary 3.3. *Let X be a homogeneous continuum, and $\mathcal{F}$ a closed, partly intrinsic collection of continua in X satisfying property* (1). *If $\mathcal{K}$ and $\mathcal{L}$ are different components of $\mathcal{F}$, then either $\mathcal{K} \prec \mathcal{L}$ or $\mathcal{L} \prec \mathcal{K}$. In other words, $\prec$ is a strict order in the collection of components of $\mathcal{F}$. If $\mathcal{F} = \mathcal{T}(X)$, then, in this order, $\mathcal{T}_0(X)$ and $\mathcal{T}_1(X)$ are the smallest and largest components of $\mathcal{F}$, respectively.*

Proof. The union $\mathcal{K} \cup \mathcal{L}$ is a compact collection, and thus it has a minimal member Y. The continuum Y is in exactly one of the collections $\mathcal{K}$ or $\mathcal{L}$, say $Y \in \mathcal{K}$. (The proof for $Y \in \mathcal{L}$ is similar.) Since $\bigcup \mathcal{L} = X$ by Proposition 3.1, there is a $Z \in \mathcal{L}$ such that $Y \cap Z \neq \emptyset$. We have $Z \neq Y$ because $\mathcal{K}$ and $\mathcal{L}$ are disjoint. Thus $Z \not\subset Y$ because Y is minimal. Consequently, $Y \subset Z$ by property (1). By Proposition 3.2 every member of $\mathcal{K}$ is contained in some member $\mathcal{L}$.

Suppose a continuum $L \in \mathcal{L}$ is contained in some member K of $\mathcal{K}$. By Proposition 3.2, every member of $\mathcal{L}$ is contained in some member of $\mathcal{K}$. By compactness, $\mathcal{L}$ has a maximal member L_0, which is contained in some $K_0 \in \mathcal{K}$. We have shown, however, that $K_0 \subset L_1$ for some $L_1 \in \mathcal{L}$. Since L_0 is maximal, it follows $L_0 = L_1 = K_0$. Therefore K_0 is in the intersection of two different components, $\mathcal{K}$ and $\mathcal{L}$, which is impossible. Hence no member of $\mathcal{L}$ is contained in a member of $\mathcal{K}$, which completes the proof. $\square$

Corollary 3.4. *Let X be a homogeneous continuum, and $\mathcal{F}$ a closed, partly intrinsic collection of continua in X satisfying property* (1). *If X and some singleton are members of the same component of $\mathcal{F}$, then $\mathcal{F}$ is connected.*

Proof. Let $\mathcal{K}$ be the component of $\mathcal{F}$ having X and a singleton $\{p\}$ as members, and suppose there is another component $\mathcal{L}$. Since $L \subset X \in \mathcal{K}$ for $L \in \mathcal{L}$, we have $\mathcal{L} \prec \mathcal{K}$ by Corollary 3.3. In particular, no member of $\mathcal{K}$ is contained in some member of $\mathcal{L}$. Since $\bigcup \mathcal{L} = X$ by Proposition 3.1, a member of $\mathcal{K}$, $\{p\}$, is contained in some member of $\mathcal{L}$, a contradiction. $\square$

Proposition 3.5. *Let X be a homogeneous continuum, and $\mathcal{F}$ a closed, partly intrinsic collection of continua in X satisfying property* (1)*. If a non-empty collection $\mathcal{G} \subset \mathcal{F}$ is open-closed in $\mathcal{F}$, then the collections $\mathcal{G}_{\min}$ and $\mathcal{G}_{\max}$ of the minimal and maximal members of $\mathcal{G}$, respectively, form partly intrinsic, continuous, homogeneous decompositions of X. In particular, the collections $\mathcal{G}_{\min}$ and $\mathcal{G}_{\max}$ are closed in $C(X)$.*

Proof. Let $U_{\mathcal{F}} \subset H(X)$ be the neighborhood of the identity composed of the self-homeomorphisms of X that respect $\mathcal{F}$. Since $\mathcal{G}$ is open-closed in $\mathcal{F}$, there is a neighborhood $U_{\mathcal{G}} \subset U_{\mathcal{F}}$ of the identity that respects $\mathcal{G}$. Thus $\mathcal{G}$ is a closed, partly intrinsic collection, and $\bigcup \mathcal{G} = X$ by Proposition 3.1. Therefore, minimal and maximal members of $\mathcal{G}$ exist. The self-homeomorphisms of X that respect $\mathcal{G}$ also preserve $\mathcal{G}_{\min}$ and $\mathcal{G}_{\max}$, and thus $\mathcal{G}_{\min}$ and $\mathcal{G}_{\max}$ are partly intrinsic collections. Clearly, $\bigcup \mathcal{G}_{\max} = X$. If $\{x_n\} \subset X$ converges to $x_0 \in X$, by the Effros theorem for almost all n there are homeomorphisms h_n in $U_{\mathcal{G}}$ such that $h(x_0) = x_n$. Using these homeomorphisms we observe that if x_0 is in some member of $\mathcal{G}_{\min}$, then so are almost all x_n's. Similarly, if infinitely many x_n's are in some members of $\mathcal{G}_{\min}$, so is x_0. This proves that $\bigcup \mathcal{G}_{\min}$ is open-closed in X. Hence $\bigcup \mathcal{G}_{\min} = X$.

Since $\mathcal{G}_{\min}$ and $\mathcal{G}_{\max}$ satisfy property (1), they both partition X. Hence $\mathcal{G}_{\min}$ and $\mathcal{G}_{\max}$ are homogeneous and continuous decompositions by Proposition 2.4. □

In this section, we use the following definition for arbitrary closed, intrinsic family $\mathcal{F} \subset C(X)$. The case $\mathcal{F} = \mathcal{T}(X)$ will be explored in the next section.

Definition 3.6. Let $\mathcal{F}$ be a collection of continua in a space X. A continuum $C \in \mathcal{F}$ is called a *locally minimal* (*locally maximal*) in $\mathcal{F}$ provided for some neighborhood $\mathcal{W}$ of C in $C(X)$ and every $C' \in \mathcal{W} \cap \mathcal{F}$, if $C' \subset C$ (if $C \subset C'$), then $C' = C$. If $\mathcal{F} = \mathcal{T}(X)$, we call such C a *locally minimal* (*locally maximal*) *terminal* continuum in X.

Proposition 3.7. *Let X be a homogeneous continuum, and $\mathcal{F}$ a partly intrinsic collection of continua in X satisfying property* (1)*. Then every locally minimal* (*locally maximal*) *member of $\mathcal{F}$ is homogeneous and partly fastened in X.*

Proof. Let $U_0 \subset H(X)$ be a neighborhood of the identity composed of the homeomorphisms that respect $\mathcal{F}$, and Y a locally minimal member of $\mathcal{F}$ with a neighborhood $\mathcal{W}$ of Y in $C(X)$, as in the definition. For some neighborhood $U_1 \subset H(X)$ of the identity and every $h \in U_1$ we have $h(Y) \in \mathcal{W}$. Define $U = U_0 \cap U_1 \cap U_0^{-1} \cap U_1^{-1}$, and note that U is a symmetric neighborhood of the identity, that is, $h \in U$ if and only if $h^{-1} \in U$. Let $h \in U$ and assume $h(Y) \cap Y \neq \emptyset$. Clearly $h(Y), h^{-1}(Y) \in \mathcal{W}$. Either $h(Y) \subset Y$ or $Y \subset h(Y)$ by property (1). Equivalently, either $h(Y) \subset Y$ or $h^{-1}(Y) \subset Y$, and thus either $h(Y) = Y$ or $h^{-1}(Y) = Y$ by the definition of $\mathcal{W}$. Hence $h(Y) = Y$, and Y is partly fastened in X. It is homogeneous by Proposition 2.8. The proof for a locally maximal Y is similar. □

In the next result we use a Whitney map, that is, a continuous function $\omega : C(X) \to [0,1]$ such that: (i) $\omega(\{x\}) = 0$ for each $x \in X$, (ii) $\omega(X) = 1$, and (iii) if $K \neq L$ and $K \subset L$, then $\omega(K) < \omega(L)$. It is known that every non-degenerate continuum admits a Whitney map. See [19] for an extensive study of Whitney maps.

Proposition 3.8. *Let X be a compact space, $\mathcal{F}$ a closed collection of continua in X satisfying property* (1)*, and Y a member of $\mathcal{F}$ such that $\{y\} \in \mathcal{F}$ for each $y \in Y$. If either Y contains no non-degenerate locally minimal member of $\mathcal{F}$, or Y contains no locally maximal member of $\mathcal{F}$ different from Y, then there is an order arc in $\mathcal{F}$ from some singleton $\{y_0\} \subset Y$ to Y.*

Proof. Let $\mathcal{F}_Y = \mathcal{F} \cap C(Y)$ and $\omega : C(X) \to [0,1]$ be a Whitney map. We show the conclusion in the case no proper subcontinuum of Y is locally maximal in $\mathcal{F}$. The proof in the other case is similar. Given $\varepsilon > 0$, let $\mathcal{C}_\varepsilon$ be the collection of closed collections $\mathcal{K} \subset \mathcal{F}_Y$ such that:

(a) $\bigcap \mathcal{K} = \{x\}$ for some $x \in X$, and
(b) for every $K \in \mathcal{K}$ there is a finite sequence $K_1, \ldots, K_n \in \mathcal{K}$ such that $K_1 = \{x\}$, $K_n = K$, and the Hausdorff distance from K_i to K_{i+1} is less than or equal to ε.

Note that such $\mathcal{K}$ is strictly ordered by inclusion because $Z_1 \cap Z_2 \neq \emptyset$ for each $Z_1, Z_2 \in \mathcal{K}$ and $\mathcal{K} \subset \mathcal{F}$ has property (1). Since each $\mathcal{K} \in \mathcal{C}_\varepsilon$ is closed, it has a greatest member, $\max\{\mathcal{K}\}$. Clearly, the assignment $\mathcal{K} \mapsto \max\{\mathcal{K}\}$ is continuous. It is easy to see that $\mathcal{C}_\varepsilon$ is closed in $2^{C(Y)}$ for each $\varepsilon > 0$. By the compactness of $\mathcal{C}_\varepsilon$ and continuity of ω, there is a $\mathcal{K}_0 \in \mathcal{C}_\varepsilon$ having the greatest value, $\omega(\max\{\mathcal{K}_0\})$, of all $\omega(\max\{\mathcal{K}\})$ for $\mathcal{K} \in \mathcal{C}_\varepsilon$. Let $\alpha(\varepsilon)$ be that value, and suppose $\alpha(\varepsilon) < \omega(Y)$. Since $\max\{\mathcal{K}_0\}$ is a proper subcontinuum of Y, and $\max\{\mathcal{K}_0\}$ is not locally maximal in $\mathcal{F}_Y$, there is a $\widehat{K} \in \mathcal{F}_Y$ such that $\max\{\mathcal{K}_0\} \subset \widehat{K}$, $\omega(\widehat{K}) > \omega(\max\{\mathcal{K}_0\})$, and the Hausdorff distance from $\max\{\mathcal{K}_0\}$ to $\widehat{K}$ is less than ε. We observe that $\widehat{\mathcal{K}} = \mathcal{K}_0 \cup \{\widehat{K}\}$ is in $\mathcal{C}_\varepsilon$, and, we have $\omega(\max\{\mathcal{K}_0\}) < \omega(\widehat{K}) = \omega(\max\{\widehat{\mathcal{K}}\})$, a contradiction. Therefore, $\alpha(\varepsilon) = \omega(Y)$, and thus there is a member $\mathcal{K}$ of $\mathcal{C}_\varepsilon$ with $\max\{\mathcal{K}\} = Y$ for each $\varepsilon > 0$.

Let $\mathcal{C}_\varepsilon^0$ be the collection of members $\mathcal{K}$ of $\mathcal{C}_\varepsilon$ such that $\max\{\mathcal{K}\} = Y$, and note that $\mathcal{C}_\varepsilon^0$ is closed in $\mathcal{C}_\varepsilon$. Consequently, letting $\varepsilon_n = 1/n$, we note that $\mathcal{C}_{\varepsilon/n}^0$ is a nested sequence of compact subsets of $2^{C(Y)}$, and thus there is a $\mathcal{K}^0 \in \bigcap_n \mathcal{C}_{\varepsilon/n}^0$. The collection $\mathcal{K}^0$ is an order arc from some singleton $\{x_0\}$ to X, composed of members of $\mathcal{F}_Y$. The collection $\mathcal{K}^0$ is the desired order arc in $\mathcal{F}$. □

Let X be a space, and $\mathcal{F}$ a collection of continua in X having property (1). Suppose $\mathcal{D}_A$ and $\mathcal{D}_B$ are upper semi-continuous decompositions of compact sets A and B, respectively, into members of $\mathcal{F}$. We denote by $\mathcal{D}_A \triangledown \mathcal{D}_B$ the collection of maximal members of $\mathcal{D}_A \cup \mathcal{D}_B$. It is an easy exercise to show that $\mathcal{D}_A \triangledown \mathcal{D}_B$ is an upper semi-continuous decomposition of $A \cup B$, and the operation $\triangledown$ is associative for such decompositions.

Lemma 3.9. *Let $\mathcal{F}$ be a family of subcontinua of a space X such that $\mathcal{F}$ has property* (1). *Let A and B be compact subsets of X, and, $\mathcal{D}_A$ and $\mathcal{D}_B$ decompositions of A and B, respectively, into members of $\mathcal{F}$. If $\mathcal{D}_A$ is upper semi-continuous and $\mathcal{D}_B$ continuous, then* $\dim(A \cup B)/(\mathcal{D}_A \triangledown \mathcal{D}_B) \leqslant \max\{\dim A/\mathcal{D}_A, \dim B/\mathcal{D}_B\}$.

Proof. Let $\mathcal{D} = \mathcal{D}_A \triangledown \mathcal{D}_B$, and $q_A : A \to A/\mathcal{D}_A$, $q_B : B \to B/\mathcal{D}_B$, $q : A \cup B \to (A \cup B)/\mathcal{D}$ be the corresponding quotient maps. Let $\mathcal{D}_0$ be the collection of members of $\mathcal{D}_A$ containing some members of $\mathcal{D}_B$, and $\mathcal{D}_1$ the collection of members of $\mathcal{D}_A$ disjoint with B. Since $\mathcal{D}_B$ is continuous, $\mathcal{D}_B$ is a compact subset of $C(X)$. Thus, if $\{D_n\} \subset \mathcal{D}_0$, then the upper limit of $\{D_n\}$ is contained in $A_0 = \bigcup \mathcal{D}_0$. Therefore A_0 is a closed subset of $A = \bigcup \mathcal{D}_A$, and the union B_0 of the members of $\mathcal{D}_B$ contained in some member of $\mathcal{D}_A$ is closed relative to B. Clearly $A_1 = \bigcup \mathcal{D}_1 \subset A$ is an open set relative to A and $A \cup B$, and the union $B_1 = B - B_0$ of the members of $\mathcal{D}_B$ not contained in any member of $\mathcal{D}_A$ is open relative to B. Note that by the definition of q the three following pairs of sets: $q_A(A_0)$ and $q(A_0)$; $q_A(A_1)$ and $q(A_1)$; $q_B(B_1)$ and $q(B_1)$ are homeomorphic. Each of the sets A_0, A_1 and B_1 is saturated with respect to $\mathcal{D}$, and $A_0 \cup A_1 \cup B_1 = A \cup B$.

Suppose $\max\{\dim A/\mathcal{D}_A, \dim B/\mathcal{D}_B\} \leqslant n$ for some n. To complete the proof it suffices to show that $\dim(A \cup B)/\mathcal{D} \leqslant n$. Since the set $q_A(A_1)$ is open relative to $q_A(A)$, it is the countable union of compact subsets of $q_A(A)$. The sets $q_A(A_1)$ and $q(A_1)$ are homeomorphic and $\dim q_A(A) \leqslant n$, and thus $q(A_1)$ is the countable union of its compact subsets each having dimension less than or equal to n. Also $q(A_0)$ is compact and homeomorphic $q_A(A_0)$, and $q_A(A_0) \subset q_A(A)$. Consequently, $\dim q(A_0) \leqslant n$. The set $q(B_1)$ is homeomorphic to $q_B(B_1)$, which is open relative to $q_B(B)$. Therefore $q_B(B_1)$ is the countable union of compact subsets of $q_B(B)$. Since $\dim q_B(B) \leqslant n$, the set $q(B_1)$ is the countable union of compact sets each having dimension less than or equal to n. Consequently, $(A \cup B)/\mathcal{D} = q(A \cup B) = q(A_0) \cup q(A_1) \cup q(B_1)$ is the countable union of its closed subsets of dimension less than or equal to n. Hence $\dim(A \cup B)/\mathcal{D} \leqslant n$. □

By induction, Lemma 3.9 implies the following.

Corollary 3.10. *Let $\mathcal{F}$ be a family of subcontinua of a space X such that $\mathcal{F}$ has property* (1). *Let $A_1, \dots, A_n$ be compact subsets of X, and, $\mathcal{D}_1, \dots, \mathcal{D}_n$ upper semi-continuous decompositions of $A_1, \dots, A_n$, respectively, into members of $\mathcal{F}$. If all but at most one $\mathcal{D}_j$'s are continuous, then*

$$\dim(A_1 \cup \dots \cup A_n)/(\mathcal{D}_1 \triangledown \dots \triangledown \mathcal{D}_n) \leqslant \max\{\dim A_1/\mathcal{D}_1, \dots, \dim A_n/\mathcal{D}_n\}.$$

4. The terminal hyperspace of a homogeneous continuum

If X is a homogeneous continuum, then $\mathcal{T}(X)$ is a closed, non-empty intrinsic collection of continua in X satisfying property (1). Thus all results from the previous section apply to the collection $\mathcal{F} = \mathcal{T}(X)$ of terminal subcontinua of X. In this section, we use these results without repeating each time that we mean the $\mathcal{F} = \mathcal{T}(X)$ application.

The structure of the components of $\mathcal{T}(X)$ for a homogeneous one-dimensional continuum X is known. In the following theorem, we summarize some known structural information on $\mathcal{T}(X)$.

Theorem 4.1. *Let X be a homogeneous curve. Then the following statements hold*:

(a) *$\mathcal{T}_0(X)$ and $\mathcal{T}_1(X)$ are the only components of $\mathcal{T}(X)$.*
(b) *$\mathcal{T}_0(X) = \mathcal{T}_1(X)$ if and only if X is hereditarily indecomposable.*
(c) *$\mathcal{T}_0(X) \neq \mathcal{T}_1(X)$ if and only if $\mathcal{T}_1(X)$ equals the singleton $\{X\}$.*
(d) *Each subcontinuum of a member of $\mathcal{T}_0(X)$ is in $\mathcal{T}_0(X)$.*
(e) *All members of $\mathcal{T}_0(X)$ are hereditarily indecomposable, tree-like continua.*

Proof. If X is tree-like, then X is hereditarily indecomposable by [11], and hence every subcontinuum of X is terminal. In particular $\mathcal{T}(X) = C(X)$ is connected, and thus $\mathcal{T}(X)$ has only one component in this case. If X is not tree-like, the terminal decomposition $\Phi(X)$ is composed of proper, terminal, tree-like, homogeneous subcontinua Y of X [27]. Again by [11], each such Y has all subcontinua terminal, which are in $\mathcal{T}_0(X)$. The quotient $X/\Phi(X)$ has no proper, non-degenerate terminal subcontinua. Therefore, all proper terminal subcontinua of X that are saturated with respect to $\Phi(X)$, are in $\Phi(X)$. Thus we have a separation of $\mathcal{T}(X)$ into $\mathcal{T}_1(X) = \{X\}$ and $\mathcal{T}_0(X)$ composed of the continua contained in some members of $\Phi(X)$. In particular, X is not hereditarily indecomposable in this case. Note that all parts of the theorem hold in both cases. □

Observe the following corollary, which is an immediate consequence of Theorem 4.1. Indeed, A_0 and A_1 in this corollary, if different, can only be members of $\mathcal{T}_0(X)$ by parts (a) and (c) of Theorem 4.1. Thus A_1 has all subcontinua terminal by part (e), and each order arc from A_0 to A_1 (in fact such an arc is unique) is contained in $\mathcal{T}(X)$.

Corollary 4.2. *If X is a homogeneous curve, A_0, A_1 are terminal subcontinua of X in the same component of $\mathcal{T}(X)$, and $A_1 \subset A_2$, then there is an order arc in $\mathcal{T}(X)$ from A_0 to A_1.*

To determine structural properties of $\mathcal{T}(X)$ for a homogeneous continuum X of arbitrary dimension, is the main objective of the study presented in this section. We begin with the following key result.

Theorem 4.3. *If X is a homogeneous continuum, then $\mathcal{T}(X)$ has at most three components. Moreover, if $\mathcal{K}$ is a component of $\mathcal{T}(X)$, and $\mathcal{K}_{\min}$, $\mathcal{K}_{\min}$ the collections of its minimal and maximal members, respectively, then $\mathcal{K}$ is intrinsic in X, and $\mathcal{K}_{\min}$ and $\mathcal{K}_{\min}$ are intrinsic, continuous, homogeneous decompositions of X.*

Proof. Suppose $\mathcal{T}(X)$ has at least four components. By Corollary 3.3 there are components P, Q, R, S of $\mathcal{T}(X)$ such that $P \prec Q \prec R \prec S$. Consequently, there are mutually disjoint open-closed collections $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$ in $\mathcal{T}(X)$ such that $\mathcal{P} \prec \mathcal{Q} \prec \mathcal{R} \prec \mathcal{S}$

Let $\mathcal{Q}_{\min}$ be the collection of the minimal members of $\mathcal{Q}$. By Proposition 3.5, $\mathcal{Q}_{\min}$ is a partly intrinsic, atomic, continuous, homogeneous decomposition of X. Therefore, $Y = X/\mathcal{Q}_{\min}$ is homogeneous. Each member Z of $\mathcal{Q}_{\min}$ is non-degenerate because it properly contains some members of $\mathcal{P}$, and, Z is a proper subcontinuum of X because it is properly contained in some member of $\mathcal{R}$. By Theorem 2.15, Y is a curve, and thus $\mathcal{T}(Y)$ has at most two components by Theorem 4.1(a). On the other hand, by Proposition 2.12, $\mathcal{T}(Y)$ is isomorphic to the collection $\mathcal{F}_0$ of the members of $\mathcal{T}(X)$ saturated with respect to $\mathcal{Q}_{\min}$. The collection $\mathcal{F}_0$ contains at least three open-closed, mutually disjoint subsets, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$. Thus $\mathcal{F}_0$ has at least three components. Hence $\mathcal{T}(Y)$ has at least three components, a contradiction.

Since there are at most three components of $\mathcal{T}(X)$, and the order $\prec$ is respected by the self-homeomorphisms of X, each component of $\mathcal{T}(X)$ is an intrinsic collection in X, and so are its minimal and minimal members. The remaining part of the conclusion follows from Proposition 3.5. $\square$

Proposition 4.4. *If X is a homogeneous continuum, and $\mathcal{T}(X)$ has exactly three components, then* $\dim X > 1$, *the component $\mathcal{T}_0(X)$ is composed only of the singletons $\{x\}$ for $x \in X$, and $\mathcal{T}_1(X)$ equals the singleton $\{X\}$.*

Proof. Assume that $\mathcal{T}(X)$ has exactly three components, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. We have $\dim X > 1$ by Theorem 4.1(a). By Corollary 3.3 the relation $\prec$ is a strict order in $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$, say $\mathcal{A} \prec \mathcal{B} \prec \mathcal{C}$. Clearly, $\mathcal{A} = \mathcal{T}_0(X)$, and $\mathcal{C} = \mathcal{T}_1(X)$. The collection $\mathcal{B}$ is an open-closed subset of $\mathcal{T}(X)$ composed of non-degenerate, proper subcontinua of X. The collection $\mathcal{B}_{\min}$ of the minimal members of $\mathcal{B}$ is a partly intrinsic, atomic, continuous, homogeneous decomposition of X by Proposition 3.5. Therefore, $Y = X/\mathcal{B}_{\min}$ is homogeneous, and moreover, Y is a curve by Theorem 2.15. By Proposition 2.12, the terminal hyperspace $\mathcal{T}(Y)$ is isomorphic to the collection $\mathcal{F}_0$ of the continua in $\mathcal{T}(X)$ saturated with respect to $\mathcal{B}_{\min}$, and $\mathcal{F}_0$ contains exactly two components, $\mathcal{C} = \mathcal{T}_1(X)$ and $\mathcal{B}$. Thus $\mathcal{T}_0(Y) \neq \mathcal{T}_1(Y)$. By Theorem 4.1(c) we have $\mathcal{T}_1(Y) = \{Y\}$. The isomorphism between $\mathcal{F}_0$ and $\mathcal{T}(Y)$ transforms $\mathcal{T}_1(X)$ to $\mathcal{T}_1(Y)$. Hence $\mathcal{T}_1(X)$ is a singleton.

Suppose $\mathcal{A} = \mathcal{T}_0(X)$ has non-degenerate continua as members. The collection $\mathcal{A}$ is an open-closed subset of $\mathcal{T}(X)$, and thus the collection $\mathcal{A}_{\max}$ of the maximal members of $\mathcal{A}$ is a partly intrinsic, atomic, continuous, homogeneous decomposition of X by Proposition 3.5. By assumption the members of $\mathcal{A}_{\max}$ are non-degenerate subcontinua of X, and, they are proper subcontinua of X because $\mathcal{A} \prec \mathcal{B}$. The quotient $Z = X/\mathcal{A}_{\max}$ is homogeneous, and Z is a curve by Theorem 2.15. As in the previous case, using Proposition 2.12 we argue that $\mathcal{T}(Z)$ is isomorphic to the collection $\mathcal{G}$ of terminal subcontinua of X saturated with respect to $\mathcal{A}_{\max}$. The collection $\mathcal{G}$ has three components, $\mathcal{A}_{\max}$, $\mathcal{B}$ and $\mathcal{C} = \mathcal{T}_1(X)$. Hence $\mathcal{T}(Z)$ has three components, which is impossible by Theorem 4.1(a). The proof is complete. $\square$

Proposition 4.5. *Let X be a homogeneous continuum such that the collection $\mathcal{T}_1(X)$ has more than one member. Then $\mathcal{T}(X)$ has at most two components. Moreover, if $\mathcal{T}(X)$ has exactly two components, then* $\dim X > 1$, *and the component $\mathcal{T}_0(X)$ is composed only of the singletons $\{x\}$ for $x \in X$.*

Proof. The first part of the conclusion immediately follows from Proposition 4.4. The argument showing the second part resembles the one used in the proof of Proposition 4.4. Assume $\mathcal{T}(X)$ has exactly two components, $\mathcal{T}_0(X)$ and $\mathcal{T}_1(X)$, and suppose $\mathcal{T}_0(X)$ has non-degenerate members. The collection $\mathcal{A}_{\max}$ of the maximal members of $\mathcal{T}_0(X)$ is a partly intrinsic, atomic, continuous, homogeneous decomposition of X by Proposition 3.5, and $\mathcal{A}_{\max}$ is composed of proper, non-degenerate continua. The quotient $Z = X/\mathcal{A}_{\max}$ is homogeneous, and Z is a curve by Theorem 2.15. Further, $\mathcal{T}(Z)$ is isomorphic to the collection $\mathcal{G}$ of terminal subcontinua of X saturated with respect to $\mathcal{A}_{\max}$. The collection $\mathcal{G}$ has exactly two components, $\mathcal{A}_{\max}$ and $\mathcal{T}_1(X)$, and $\mathcal{T}_1(X)$ is non-degenerate. Thus $\mathcal{T}_1(Z)$ is non-degenerate. By Theorem 4.1(c), the component $\mathcal{T}_1(Z)$ is degenerate, a contradiction. This shows that $\mathcal{T}_0(X)$ is composed only of singletons. The dimension of X cannot be one by Theorem 4.1(c), and hence $\dim X > 1$. $\square$

From Propositions 4.4 and 4.5 we observe the following.

Corollary 4.6. *If X is a homogeneous continuum, then at most one component $\mathcal{K}$ of $\mathcal{T}(X)$ can have members that are neither minimal nor maximal in $\mathcal{K}$.*

Let X be a homogeneous continuum, $\mathcal{A}$ a component of $\mathcal{T}(X)$, and $\mathcal{A}_{\min}$, $\mathcal{A}_{\max}$ the collections of minimal and maximal members of $\mathcal{A}$, respectively. Every $p \in X$ is exactly in one continuum $A_0 \in \mathcal{A}_{\min}$, and in one $A_1 \in \mathcal{A}_{\max}$. Now we focus on the following problem. Must there be an order arc in $\mathcal{T}(X)$ from A_0 to A_1?

If $\dim X \leqslant 1$, we have a positive answer to this question by Corollary 4.2. To examine this problem for higher dimensional spaces, we use *locally minimal* and *locally maximal terminal continua* introduced in Definition 3.6. In the remaining part of this paper, we frequently use the following observation, which is an easy consequence of Proposition 2.13.

Proposition 4.7. *Let Y be a terminal subcontinuum of a continuum X. Then a proper subcontinuum Z of Y is a locally minimal (maximal) terminal subcontinuum of Y if and only if Z is a locally minimal (maximal) terminal subcontinuum of X.*

Proposition 4.8. *If a terminal subcontinuum Y of a homogeneous continuum X has no non-degenerate locally minimal terminal subcontinua, then Y is hereditarily indecomposable, tree-like continuum.*

Proof. By Proposition 3.8 there is an order $\mathcal{K}$ arc of terminal subcontinua of X from a singleton $\{y\} \subset Y$ to Y. The collection $\mathcal{C}$ of order arcs in $\mathcal{T}(X)$ having some singleton as its one end-point is closed as a subspace of $2^{\mathcal{T}(X)}$. Therefore $\bigcup \mathcal{C}$ is closed in $\mathcal{T}(X)$. Thus there is a maximal member Y_0 of $\bigcup \mathcal{C}$ containing Y. The continuum Y_0 is fastened. Indeed, if $h : X \to X$ be a homeomorphism such that $h(Y_0) \cap Y_0 \neq \emptyset$, then either $Y_0 \subset h(Y_0)$ or $Y_0 \subset h^{-1}(Y_0)$ by property (1). Since the collection of maximal members of $\bigcup \mathcal{C}$ is intrinsic, in both cases we have $h(Y_0) = Y_0$. Thus Y_0 is homogeneous by Proposition 2.8.

Let $\omega : C(X) \to [0,1]$ be a Whitney map, and $\mathcal{K}_0$ an order arc in $\mathcal{T}(X)$ from some singleton $\{y_0\} \subset Y_0$ to Y_0. Let $Z \in C(Y_0)$. Since Y_0 is fastened and X homogeneous, without loss of generality assume $y_0 \in Z$. We have $\omega(Z) \leqslant \omega(Y_0)$, and thus there is a continuum $K \in \mathcal{K}_0$ such that $\omega(K) = \omega(Z)$. Note neither of the continua K and Z can be properly contained in the other because $\omega(K) = \omega(Z)$. Since $y_0 \in K \cap Z \neq \emptyset$, $K = Z$ by the terminality of K. Thus Z is terminal. Since every subcontinuum of Y_0 is terminal, Y_0 is hereditarily indecomposable. Also, Y_0 is tree-like by [24]. Since Y_0 is hereditarily indecomposable and tree-like, so is its subcontinuum, Y. $\square$

Theorem 4.9. *If a homogeneous continuum X has arbitrarily small, non-degenerate terminal subcontinua, then* $\dim X = 1$. *Equivalently, if* $\dim X > 1$, *then $\mathcal{T}_0(X)$ is composed only of singletons.*

Proof. Let Z_n be non-degenerate terminal subcontinua of X such that $\lim \operatorname{diam} Z_n = 0$. If a Z_n has no non-degenerate locally minimal terminal subcontinuum, then it is hereditarily indecomposable and tree-like by Proposition 4.8, and thus every subcontinuum of Z_n is terminal in X. By the homogeneity and compactness of X, and the terminality of Z_n, all sufficiently small subcontinua of X are contained in some $h(Z_n)$ for $h \in H(X)$. Thus all sufficiently small subcontinua of X are tree-like, and hence $\dim X = 1$.

Assume every Z_n has a non-degenerate, locally minimal terminal subcontinuum Y_n. Without loss of generality assume that Y_n's converge in $C(X)$. Since $\lim \operatorname{diam} Y_n = 0$, Y_n's limit on a singleton $\{x_0\}$ in X. By the Effros theorem there are homeomorphisms $h_n : X \to X$ converging to the identity such that $h_n(x_0) \in Y_n$. We have $\operatorname{Lim} h_n^{-1}(Y_n) = \bigcap h_n^{-1}(Y_n) = \{x_0\}$. By the homogeneity of X, every point in X is in arbitrarily small, non-degenerate, locally minimal, terminal subcontinua of X.

To show that X is one-dimensional, we fix an $\varepsilon > 0$ and prove there exists an ε-map $f_\varepsilon : X \to M$, where M is a one-dimensional continuum.

Fix $x \in X$, and a non-degenerate, locally minimal terminal continuum Y_x such that $x \in Y_x$ and $\operatorname{diam} Y_x < \varepsilon$. By Proposition 3.7, Y_x is homogeneous, and Y_x is not approximated by its proper terminal subcontinua. Let $\Phi(Y_x)$ be Rogers' terminal decomposition of Y_x (see Section 2 for definition). Thus $\Phi(Y_x)$ is composed of maximal proper terminal subcontinua of Y_x, and it is intrinsic in Y_x. By Proposition 2.6, choose an open neighborhood U_x of the identity in $H(X)$ such that $\operatorname{diam} \bigcup \{h(Y_x)\colon\ h \in U_x\} < \varepsilon$, and, if $g, h \in U_x$, then $g(Y_x) = h(Y_x)$. Let $\mathcal{W}(x) = \{h(Y_x) \mid h \in U_x\}$. By the Effros theorem the union $W_x = \bigcup \mathcal{W}(x)$ is an open neighborhood of Y_x. The collection $\mathcal{W}(x)$ partitions W_x by the definition of U_x. Since $\Phi(Y_x)$ is intrinsic, for every $h \in U_x$ we have $\{h(Z) \mid Z \in \Phi(Y_x)\} = \Phi(h(Y_x))$. Thus the collection $\mathcal{V}(x) = \{h(Z) \mid h \in U_x \text{ and } Z \in \Phi(Y_x)\}$ also partitions W_x, and its members are properly contained in the members of $\mathcal{W}(x)$.

Claim. *Both $\mathcal{W}(x)$ and $\mathcal{V}(x)$ are continuous decompositions of W_x.*

Indeed, let a sequence $\{p_n\} \subset W_x$ converge to some $p \in W_x$, and $h(Y_x)$ be the member of $\mathcal{W}(x)$ containing p for some $h \in U_x$. Let $h_n : X \to X$ be homeomorphisms converging to the identity such that $h_n(p) = p_n$. Then for almost all n's, the composition $h_n \circ h$ is in U_x, and thus $h_n(h(Y_x))$ is the member of $\mathcal{W}(x)$ containing p_n. We have $\operatorname{Lim} h_n(h(Y_x)) = h(Y_x)$, which shows the continuity of $\mathcal{W}(x)$. Since the homeomorphisms h and h_n, for almost all n, respect the decomposition $\mathcal{V}(x)$, the continuity of $\mathcal{V}(x)$ also follows.

By the Claim, the quotient map $q_x : W_x \to W_x/\mathcal{W}(x)$ is open, and $W_x/\mathcal{W}(x)$ locally compact. Let P_x be a compact neighborhood of $q_x(x)$ in $W_x/\mathcal{W}(x)$, and define $Q_x = q_x^{-1}(P_x)$. We observe that Q_x is a compact subset of W_x saturated with respect to both $\mathcal{W}(x)$ and $\mathcal{V}(x)$. Let $\mathcal{Q}(x) = \mathcal{V}(x) \cap C(Q_x)$, note that $\mathcal{Q}(x)$ is a continuous decomposition of Q_x, and let $\hat{q}_x : Q_x \to Q_x/\mathcal{Q}(x)$ be the quotient map. If $Y \in \mathcal{W}(x) \cap C(Q_x)$ and $Y \subset Q_x$, then $\{Z \in \mathcal{Q}(x) \mid Z \subset Y\}$ is Rogers' terminal decomposition $\Phi(Y)$ of Y. Thus $\hat{q}_x(Y)$ is one-dimensional by Theorem 2.15. Since the continua $\hat{q}_x(Y)$, for $Y \in \mathcal{W}(x) \cap C(Q_x)$, are terminal and non-degenerate in $Q_x/\mathcal{Q}(x)$, they contain all sufficiently small subcontinua of $Q_x/\mathcal{Q}(x)$. Thus $\dim Q_x/\mathcal{Q}(x) = 1$.

As above, for every $x \in X$ we define a compact neighborhood Q_x of x in X, and a continuous decomposition $\mathcal{Q}(x)$ of Q_x into terminal subcontinua of X such that $\operatorname{diam} Q_x < \varepsilon$ and $\dim(Q_x/\mathcal{Q}(x)) = 1$. Since X is compact, $X = Q_{x_1} \cup \cdots \cup Q_{x_k}$ for some finite set $\{x_1, \dots, x_k\} \subset X$. By Corollary 3.10, the collection $\mathcal{D} = \mathcal{Q}(x_1) \triangledown \cdots \triangledown \mathcal{Q}(x_k)$ is an upper semi-continuous decomposition of X such that

$$\dim X/\mathcal{D} \leqslant \max\{Q_{x_1}/\mathcal{Q}(x_1), \dots, Q_{x_k}/\mathcal{Q}(x_k)\} = 1.$$

Each member of $\mathcal{D}$ has diameter less than ε, and thus the quotient map of this decomposition is an ε-map onto a one-dimensional continuum $M = X/\mathcal{D}$. Hence $\dim X = 1$. □

Next we show a "higher dimensional terminal decomposition theorem" for homogeneous continua. As other intrinsic decomposition theorems, such as the aposyndetic [8], terminal [27], mutually aposyndetic [20] and semi-terminal [23] ones, it reveals important structural information on homogeneous continua. Nevertheless, the reader should be reminded that the decomposition $\mathcal{J}(X)$ in this theorem can be non-trivial and have non-degenerate members only if X is a continuum for which Question 1 has a negative answer. Since no such X is known so far, this theorem can also be viewed as partial step towards answering Question 1.

Theorem 4.10. *For every homogeneous continuum X there exists a finest upper semi-continuous decomposition, $\mathcal{J}(X)$, of X into terminal subcontinua of X such that $\dim X/\mathcal{J}(X) \leqslant 1$. The decomposition $\mathcal{J}(X)$ is intrinsic, continuous and homogeneous. The members of $\mathcal{J}(X)$ have no proper non-degenerate terminal subcontinua, and if $\dim X > 1$, their dimension is the same as that of X. If $\mathcal{J}(X)$ is a non-trivial decomposition, the members of $\mathcal{J}(X)$ are indecomposable and cell-like.*

Proof. If $\dim X \leqslant 1$, letting $\mathcal{J}(X) = \{\{x\} \mid x \in X\}$ the conclusion clearly holds. Assume $\dim X > 1$, and note that X has no arbitrarily small terminal subcontinua by Theorem 4.9. Thus the collection $\mathcal{F}$ of non-degenerate terminal subcontinua of X is compact and open in $\mathcal{T}(X)$. Let $\mathcal{J}(X)$ be the collection of minimal members of $\mathcal{F}$. Then $\mathcal{J}(X)$ is an intrinsic decomposition of X by Proposition 3.5, and consequently, it is continuous and homogeneous by Proposition 2.4. The quotient $X/\mathcal{J}(X)$ has dimension at most 1 by Theorem 2.15. Since all sufficiently small subcontinua of X are contained in members of $\mathcal{J}(X)$, the dimension of the members of $\mathcal{J}(X)$ equals $\dim X$. If $\mathcal{D}$ in an upper semi-continuous decomposition of X into terminal subcontinua such that $\mathcal{J}(X)$ is not finer than $\mathcal{D}$, then $\mathcal{D}$ has all singletons in some open set W as members. Such W must contain higher dimensional continua by the homogeneity of X, and thus the quotient would have dimension greater than 1. If $\mathcal{J}(X)$ is non-trivial, the members of $\mathcal{J}(X)$ are indecomposable and cell-like again by Theorem 2.15. □

By Theorem 4.10 we have the following generalization of Corollary 4.2.

Corollary 4.11. *If X is a homogeneous continuum, A_0 and A_1 are terminal subcontinua of X in the same component of $\mathcal{T}(X)$, and $A_1 \subset A_2$, then there is an order arc in $\mathcal{T}(X)$ from A_0 to A_1.*

Proof. If $\dim X = 1$, the conclusion holds by Corollary 4.2. For $\dim X > 1$ the continua A_0 and A_1, if different, cannot be contained in $\mathcal{T}_0(X)$ by Theorem 4.9, and thus they are saturated with respect to $\mathcal{J}(X)$ (see Theorem 4.10). The collection of terminal subcontinua of X saturated with respect to $\mathcal{J}(X)$ is isomorphic to $\mathcal{T}(X/\mathcal{J}(X))$ (see Section 2). Since $X/\mathcal{J}(X)$ is homogeneous and $\dim X/\mathcal{J}(X) \leqslant 1$, the conclusion follows by Corollary 4.2. □

Corollary 4.12. *Every locally minimal* (*locally maximal*) *terminal subcontinuum of a homogeneous continuum X is a minimal* (*maximal*) *member of some component of $\mathcal{T}(X)$.*

In the following corollary, which characterizes hereditarily indecomposable homogeneous continua, we combine new and previously known information. The implications (c) ⇒ (b) and (b) ⇒ (a) in this corollary, are new.

Corollary 4.13. *For every non-degenerate homogeneous continuum X the following conditions are equivalent*:

(a) *X is hereditarily indecomposable*;
(b) *$\mathcal{T}(X)$ is connected*;

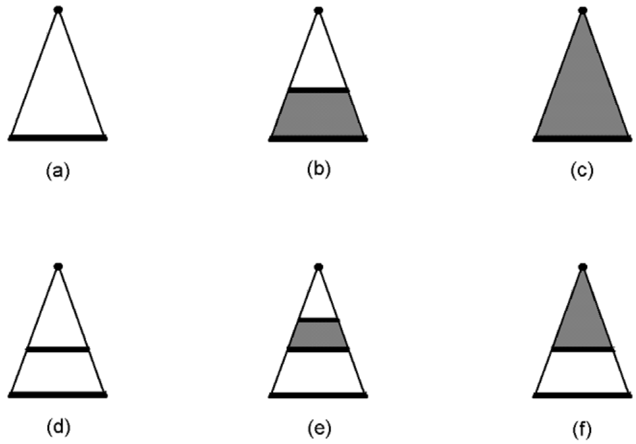

Fig. 1.

(c) *Both X and the singletons are the limits of some proper non-degenerate terminal subcontinua of X; and*
(d) *X is tree-like.*

Proof. The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious. If the singletons are limits of non-degenerate terminal continua, then $\dim X = 1$ by Theorem 4.9. Thus $\mathcal{T}(X)$ has only one component by Theorem 4.1(a), (c), and consequently X is tree-like by Theorem 4.1(e). Thus (c) $\Rightarrow$ (d) holds. If X is tree-like, then X is hereditarily indecomposable by [11]. $\square$

Comment. The original proof showing that terminal proper subcontinua of homogeneous continua are indecomposable is due to Maćkowiak and Tymchatyn [16, (6.13), p. 18]. A different proof of this last statement can also be found in [21]. The assertion that proper terminal subcontinua of homogeneous continua are cell-like was published in [31, Theorem 11, p. 420]. However, there is a question about the completeness of its proof (compare the review of this article in Zentralblatt, Zbl 1006.54042, by A. Koyama). The authors clearly use Rogers' terminal decomposition theorem, which was only proved for decomposable continua, and one-dimensional continua X with $H^1(X) \neq 0$ [27]. This limits the validity of their argument. Below we provide a complete proof of this assertion, together with another proof that proper terminal subcontinua of homogeneous continua are indecomposable. Both proofs are based on the theory developed in this paper. This result generalizes Roger's Theorem 2.15.

Theorem 4.14. *Every proper terminal subcontinuum of a homogeneous continuum is indecomposable and cell-like.*

Proof. Let Y be a proper terminal subcontinuum of a homogeneous continuum X. If Y is a singleton, the conclusion is trivial. Otherwise, Y is saturated with respect to $\mathcal{J}(X)$ (see Theorem 4.10). Let $q : X \to X/\mathcal{J}(X)$ be the quotient map of $\mathcal{J}(X)$. Since $q(Y)$ is a proper terminal subcontinuum of a homogeneous curve $X/\mathcal{J}(X)$, by Theorem 4.1(e), it is indecomposable. The map q is atomic, and thus Y is indecomposable by Proposition 2.11.

By Theorem 4.1(e), the continuum $q(Y)$ is tree-like. The map $q : X \to X/\mathcal{J}(X)$ has cell-like fibers $q^{-1}(y)$ and $\dim X/\mathcal{J}(X) = 1$ by Theorem 2.15. By [17, Theorem 7, p. 286], the map q is a hereditary shape equivalence (see [17] for the definition). Consequently, since $q(Y)$ is cell-like, so is $q^{-1}(q(Y)) = Y$. $\square$

Description of Fig. 1. Let X be a non-degenerate, homogeneous continuum. The terminal hyperspace $\mathcal{T}(X)$ of such X must have one of six types. Fig. 1 schematically represents these types. The whole triangle stands for the hyperspace $C(X)$ of subcontinua of X with the top vertex representing X as a member of $C(X)$, and the bottom edge the collection $\{\{x\} \mid x \in X\}$ of singletons in X. The reader can visualize the horizontal projection of the whole triangle on a vertical line as a Whitney map $\omega : C(X) \to [0, 1]$ even though not every Whitney map has to agree with this visualization. The horizontal thick segments and shaded areas represent the hyperspace $\mathcal{T}(X)$ of terminal subcontinua of X. The thick segments and the top vertex are the sets of minimal or maximal members of the corresponding components of $\mathcal{T}(X)$.

If X is a curve, its hyperspace $\mathcal{T}(X)$ is shown in (a), (b) or (c). Its decomposition $\mathcal{J}(X)$ (see Theorem 4.10) is the bottom thick segment. Notice the following well-known examples of homogeneous curves X of the corresponding types: (a) the circle, (b) the circle of pseudo-arcs, and (c) the pseudo-arc. Picture (a) also represents higher dimensional continua without proper non-degenerate terminal subcontinua. In this case $\mathcal{J}(X) = \{X\}$ is trivial. The two-dimensional torus is a simple example of such continuum.

Pictures (d), (e) and (f) correspond to higher dimensional homogeneous continua with proper non-degenerate terminal subcontinua. We do not know whether such continua exist. In (d), (e) and (f), the second thick horizontal segment from the bottom represents the decomposition $\mathcal{J}(X)$.

References

[1] D.P. Bellamy, J.M. Łysko, Factorwise rigidity of the product of two pseudo-arcs, in: Proceedings of the 1983 Topology Conference, Houston, Tex., 1983, Topology Proc. 8 (1) (1983) 21–27.
[2] R.H. Bing, A homogeneous indecomposable plane continuum, Duke Math. J. 15 (1948) 729–742.
[3] R.H. Bing, Concerning hereditarily indecomposable continua, Pacific J. Math. 1 (1951) 43–51.
[4] R.H. Bing, F.B. Jones, Another homogeneous plane continuum, Trans. Amer. Math. Soc. 90 (1959) 171–192.
[5] J.J. Charatonik, T. Mackowiak, Around Effros' theorem, Trans. Amer. Math. Soc. 298 (1986) 579–602.
[6] E.G. Effros, Transformation groups and C^*-algebras, Ann. of Math. (2) 81 (1965) 38–55.
[7] C.L. Hagopian, Indecomposable homogeneous plane continua are hereditarily indecomposable, Trans. Amer. Math. Soc. 224 (1976) 339–350.
[8] F.B. Jones, On a certain type of homogeneous plane continuum, Proc. Amer. Math. Soc. 6 (1955) 735–740.
[9] J.L. Kelley, Hyperspaces of a continuum, Trans. Amer. Math. Soc. 52 (1942) 22–36.
[10] B. Knaster, Un continu dont tout sous-continu est indécomposable, Fund. Math. 3 (1922) 247–286.
[11] P. Krupski, J.R. Prajs, Outlet points and homogeneous continua, Trans. Amer. Math. Soc. 318 (1990) 123–141.
[12] K. Kuperberg, W. Kuperberg, W.R.R. Transue, On the 2-homogeneity of Cartesian products, Fund. Math. 110 (1980) 131–134.
[13] W. Lewis, Continuum theory problems, Topology Proc. 8 (1983) 361–394.
[14] W. Lewis, Continuous curves of pseudo-arcs, Houston J. Math. 11 (1985) 91–99.
[15] T. Maćkowiak, Terminal continua and the homogeneity, Fund. Math. 127 (1987) 177–186.
[16] T. Maćkowiak, E.D. Tymchatyn, Continuous mappings on continua II, Dissertationes Math. (Rozprawy Mat.) 225 (1984) 1–57.
[17] S. Mardešić, J. Segal, Shape Theory, North Holland Publishing Company, 1982.
[18] E.E. Moise, An indecomposable plane continuum which is homeomorphic to each of its non-degenerate subcontinua, Trans. Amer. Math. Soc. 63 (1948) 581–594.
[19] S.B. Nadler Jr., Hyperspaces of Sets, Marcel Dekker Inc., 1978.
[20] J.R. Prajs, Mutually aposyndetic decomposition of homogeneous continua, Canad. J. Math., in press.
[21] J.R. Prajs, Filament local product structures in homogeneous continua, preprint.
[22] J.R. Prajs, Semi-terminal continua in Kelley spaces, Trans. Amer. Math. Soc., in press.
[23] J.R. Prajs, Semi-terminal continua in homogeneous spaces, preprint.
[24] J.T. Rogers Jr., Homogeneous hereditarily indecomposable continua are tree-like, Houston J. Math. 8 (1982) 421–428.
[25] J.T. Rogers Jr., Decompositions of homogeneous continua, Pacific J. Math. 99 (1982) 137–144.
[26] J.T. Rogers Jr., Cell-like decompositions of homogeneous continua, Proc. Amer. Math. Soc. 87 (1983) 375–377.
[27] J.T. Rogers Jr., Decompositions of continua over the hyperbolic plane, Trans. Amer. Math. Soc. 310 (1988) 277–291.
[28] J.T. Rogers Jr., Higher dimensional aposyndetic decompositions, Proc. Amer. Math. Soc. 131 (2003) 3285–3288.
[29] G.S. Ungar, On all kinds of homogeneous spaces, Trans. Amer. Math. Soc. 212 (1975) 393–400.
[30] R.W. Wardle, On a property of J.L. Kelley, Houston J. Math. 3 (1977) 291–299.
[31] Zhou Youcheng, Lin Shou, Homogeneity, terminality and some mapping problems, Appl. Math. J. Chinese Univ. Ser. B 16 (2001) 418–422.